# Study of Classical Mechanical Systems with Complex Potentials

*A. Sinha*[1]
*Department of Applied Mathematics,*
*Calcutta University,*
*92, A.P.C. Road, Kolkata - 700 009*
*INDIA*

*D. Dutta*[2]
*Physics & Applied Mathematics Unit,*
*Indian Statistical Institute,*
*203, B.T. Road, Kolkata - 700 108*
*INDIA*

*and*

*P. Roy*[3]
*Physics & Applied Mathematics Unit,*
*Indian Statistical Institute,*
*203, B.T. Road, Kolkata - 700 108*
*INDIA*

**Abstract**

We apply the factorization technique developed by Kuru et. al. [Ann. Phys. **323** (2008) 413] to study complex classical systems. As an illustration we apply the technique to study the classical analogue of the exactly solvable $\mathcal{PT}$ symmetric Scarf II model, which exhibits the interesting phenomenon of spontaneous breakdown of $\mathcal{PT}$ symmetry at some critical point. As the parameters are tuned such that energy switches from real to complex conjugate pairs, the corresponding classical trajectories display a distinct characteristic feature — the closed orbits become open ones.

**Key words :** $\mathcal{PT}$ symmetry, classical trajectories, open orbits, closed orbits, phase space trajectories

[1]e-mail : anjana23@rediffmail.com
[2]e-mail : debjit_r@isical.ac.in
[3]e-mail : pinaki@isical.ac.in

## 1 Introduction

Fairly recently Kuru *et. al.* obtained exact solutions of the classical analogues of some exactly solvable one-dimensional quantum mechanical models in the framework of the factorization technique [1]. However, it must be kept in mind that this method cannot be applied to all classical systems as not all quantum mechanical models have exactly solvable classical analogues. On the other hand there have been several attempts to see if the spontaneous breakdown of $\mathcal{PT}$ symmetry at an *exceptional point* in a certain class of non Hermitian $\mathcal{PT}$ symmetric quantum mechanical systems gets manifested in the corresponding classical picture [2, 3, 4, 5, 6, 7, 8]. For example, the $\mathcal{PT}$ symmetric model $H = p^2 + x^2(ix)^\epsilon$, was studied numerically in ref. [2] to show that the exceptional point occurs at $\epsilon = 0$ in both the quantum as well as the corresponding classical systems. In a different work [6] the classical motion of two 1-dimensional non Hermitian systems was studied — viz., the complex Harmonic oscillator and the complex cubic potential — and an attempt was made to find a connection between the reality of the spectrum and the regularity of the classical trajectories. Motivated by these studies, in this work we intend to extend the factorization technique of ref. [1] to study complex classical systems, which are classical analogues of non Hermitian quantum mechanical systems. In particular, we shall apply the technique to study the classical analogue of the following exactly solvable quantum mechanical $\mathcal{PT}$ symmetric (complex) Scarf II potential [9]

$$V(x) = -v_1 \ \mathrm{sech}^2 x - \hat{v}_1 \ \mathrm{sech}\ x \ \tanh x \qquad ; \qquad v_1 \text{ real} \ , \ v_1 > 0 \tag{1}$$

For $\hat{v}_1$ pure imaginary (say $\hat{v}_1 = iv_2$), the model belongs to the well known category of $\mathcal{PT}$ symmetric potentials of the form $V(x) = v_1 V_R(x) + iv_2 V_I(x)$ , with $V_R(x)$ even, and $V_I(x)$ odd. It is one of the few exactly solvable quantum mechanical models which exhibits the interesting phenomenon of spontaneous breakdown of $\mathcal{PT}$ symmetry at a critical value of the coupling parameter $v_2$. Additionally, its classical analogue can be solved exactly by means of the interesting factorization method developed in ref. [1]. Our aim here is to see if the above system contains periodic or irregular trajectories, and check if the spontaneous breakdown of $\mathcal{PT}$ symmetry switching energy from real values to complex conjugate pairs, manifests itself in the corresponding classical system as well, in terms of non-periodicity of orbits or any other feature.

The organization of the paper is as follows. To make the paper self contained, in Section 2 we briefly discuss the factorization technique of ref.[1] to obtain the classical trajectories of some exactly solvable one dimensional systems, and extend the same to complex classical systems. Based on the equations obtained in Section 2, in Section 3 we obtain the equations of the classical trajectories, the classical momenta and the phase-space trajectories of a particle under the influence of the complex Scarf II potential, and plot the corresponding figures for different values of the coupling parameter $\hat{v}_1$. Finally, Section 4 is kept for conclusions and discussions.

## 2 Formalism

We start with the one dimensional classical Hamiltonian (in units $\hbar = 2m = 1$)

$$H(x,p) = p^2 + V(x) \tag{2}$$

where $x, p$ are the canonical coordinates, $V(x)$ is the potential, and the Poisson bracket of $x$ and $p$ is given as $\{x, p\} = 1$. The equations of motion of the classical particle are given by the

Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} = 2p \qquad , \qquad \dot{p} = -\frac{\partial H}{\partial x} = -V'(x) \tag{3}$$

Thus $\ddot{x} = 2\dot{p} = 2V'(x)$ which on integration gives the velocity of the particle as

$$v = \frac{dx}{dt} = \pm 2\sqrt{E - V(x)} \tag{4}$$

$E$ being the energy of the classical particle. In our discussions since the particle is moving under the influence of complex forces, time $t$ is treated as a real variable, and the path $x(t)$ traced out by the particle as well as its velocity $v$ can take complex values. The roots of the equation $E - V(x) = 0$ give the locations of the turning points, while the initial conditions determine the initial velocity of the particle. It may be mentioned that by conventional notions the velocity of the particle can take real values only. So the initial positions of the particle under a real potential lie on the real axis, in between the turning points. However, if we analytically continue into the complex plane, a particle under the influence of a complex force can move about in the complex plane. Thus any point in the complex plane may be an initial starting point for the particle.

Our aim here is to obtain exact analytical solutions of the orbits traced out by the classical particle. For this we shall mainly follow the factorization technique of ref. [1]. We assume a factorization of the Hamiltonian $H$ in the form

$$H = A^+ A^- + \gamma(H) \tag{5}$$

where unlike in usual quantum mechanical factorizations, $\gamma(H)$ may depend on $H$, and $A^\pm$ (not necessarily complex conjugate) are taken to be of the form

$$A^\pm = \mp i f(x) p + \sqrt{H} g(x) + \varphi(x) + \phi(H) \tag{6}$$

Furthermore, the functions $A^\pm$ and $H$ are assumed to define a deformed algebra with Poisson Brackets as follows :

$$\begin{aligned} \{A^\pm, H\} &= \pm i\alpha(H) A^\pm \\ \{A^+, A^-\} &= -i\beta(H) \end{aligned} \tag{7}$$

where the auxiliary functions $\alpha(H)$, $\beta(H)$ and $\phi(H)$ are expressed in terms of the powers of $\sqrt{H}$. In case the quantum version admits bound states with negative energies, $\sqrt{H}$ should be replaced by $\sqrt{-H}$. Making use of equations (3), (6) and (7) we arrive at the following expressions :

$$f(x) = \frac{2}{\alpha(H)} \left[\varphi'(x) + g'(x)\sqrt{H}\right] \tag{8}$$

$$f(x)V'(x) - 2f'(x)\left[H - V(x)\right] = \alpha(H)\left\{g(x)\sqrt{H} + \varphi(x) + \phi(H)\right] \tag{9}$$

$$\begin{aligned} \beta &= 2\sqrt{H}\left[f'(x)g(x) - f(x)g'(x)\right] - \frac{1}{\sqrt{H}} g(x)\left[2f'(x)V(x) + f(x)V'(x)\right] \\ &+ 4f'(x)\frac{\partial\phi(H)}{\partial H}\left[H - V(x)\right] - 2f(x)\left[\varphi'(x) + \frac{\partial\phi(H)}{\partial H}V'(x)\right] \end{aligned} \tag{10}$$

Now we construct two quantities of the form

$$Q^{\pm} = A^{\pm} e^{\mp i\alpha(H)t} \tag{11}$$

which are time dependent integrals of motion. Nevertheless, their total time derivative vanishes

$$\frac{dQ^{\pm}}{dt} = \{Q^{\pm}, H\} + \frac{\partial Q^{\pm}}{\partial t} = 0 \tag{12}$$

Thus

$$\mid Q^{+}Q^{-} \mid = \mid A^{+}A^{-} \mid \tag{13}$$

so that the particular values of the integrals of motion $Q^{\pm}$ and those of $A^{\pm}$ may be respectively denoted by

$$Q^{\pm} = c(E)e^{\pm i\theta_0} \tag{14}$$

$$A^{\pm} \;=\; c(E)e^{\pm i\{\theta_0 + \alpha(H)t\}} \tag{15}$$

where $\theta_0$ is determined from initial conditions, and

$$c(E) = \sqrt{E - \gamma(H)} \tag{16}$$

For $c(E)$ to be real, the expression within the square root sign must be positive. This condition gives the range of energy values for the classical particle. The solution of (14) gives the trajectories $x(t)$ and momenta $p(t)$ of the corresponding classical particle in the complex plane. We illustrate our formalism with the help of an explicit example in the next section.

# 3 Classical analogue of Complex Scarf II potential

The quantum version of the $\mathcal{PT}$-symmetric (Complex) Scarf II potential displays certain interesting features —
(i) its discrete spectrum is real below the $\mathcal{PT}$ threshold, often referred to as the Exceptional (or critical) point, whereas above it the energy values occur as complex conjugate pairs
(ii) its continuous spectrum displays spectral singularity at the critical point, where the reflection and transmission coefficients tend to diverge.
For the sake of comparison of classical trajectories with other numerical studies (below and above the critical point), in this work we shall restrict ourselves to bound states only, hence, negative energies ($E < 0$). The final expression for $V(x)$ should be of the form

$$V(x) = -\gamma_0 \,\mathrm{sech}^2\, \frac{\alpha_0 x}{2} + 2\delta \,\mathrm{sech}\, \frac{\alpha_0 x}{2} \,\tanh \frac{\alpha_0 x}{2} \tag{17}$$

The parameter $\delta$ plays a crucial role here — the potential in (17) is real for real values of $\delta$, $\mathcal{PT}$ symmetric for pure imaginary values of $\delta$, and a general complex potential (without any $\mathcal{PT}$ symmetry) for complex values of $\delta$. Our aim in this section is to find the exact equations for the classical trajectories and momenta of a particle moving in the complex plane under such a potential for different $\delta$, with special emphasis on pure imaginary values, and to see if there is any change in these classical trajectories and momenta at the onset of spontaneous breakdown

of $\mathcal{PT}$ symmetry. The form of the potential in eq. (17) above demands that in the expression for $A^{\pm}$ in (6) we take the following forms of the functions $g(x)$ , $\varphi(x)$ , $\phi(H)$ :

$$g(x) \neq 0 \qquad , \qquad \varphi(x) = 0 \qquad , \qquad \phi(H) = \frac{\delta}{\sqrt{-H}}$$

so that $A^{\pm}$ reduce to

$$A^{\pm} = \mp i f(x) p + \sqrt{-H} g(x) + \frac{\delta}{\sqrt{-H}} \tag{18}$$

Solving equations (8), (9) and (10) simultaneously gives

$$g(x) = \sinh \frac{\alpha_0 x}{2} \qquad , \qquad f(x) = \cosh \frac{\alpha_0 x}{2} \qquad , \qquad \gamma(H) = -\gamma_0 + \frac{\delta^2}{H} \tag{19}$$

So using equations (14) and (16), and the expression

$$A^+ A^- = H + \gamma_0 - \frac{\delta^2}{H} \tag{20}$$

we obtain the value of $c(E)$ as

$$c(E) = \sqrt{E + \gamma_0 - \frac{\delta^2}{E}} \tag{21}$$

Equation (21) gives the range of values for energy $E$ as $c(E)$ should be real. It is worth remembering here that we are considering the energy $E$ to be negative. We investigate the different cases in some detail below :

**Case 1 :** $\delta$ is real; $\delta = \delta_R$ (say)
This gives the real Scarf II potential, dealt with in ref. [1], with classically allowed values of energy $E$ lying in the range

$$\frac{-\gamma_0 - \sqrt{\gamma_0^2 + 4\delta_R^2}}{2} \ < \ E \ < \ 0 \tag{22}$$

**Case 2 :** $\delta$ is pure imaginary; $\delta = i\delta_I$ (say)
This is the classical analogue of the famous non Hermitian yet $\mathcal{PT}$ symmetric Scarf II potential, which exhibits the phenomenon of spontaneous $\mathcal{PT}$ symmetry breaking at the *phase transition* point, and real energy values switch to complex conjugate pairs. Straightforward algebra shows that for the corresponding classical motion $c(E)$ reduces to

$$c(E) = \sqrt{E + \gamma_0 + \frac{\delta_I^2}{E}} \tag{23}$$

Since we are dealing with negative energies, the classically allowed range for $E$ in this case becomes

$$\frac{-\gamma_0 - \sqrt{\gamma_0^2 - 4\delta_I^2}}{2} \ < \ E \ < \ \frac{-\gamma_0 + \sqrt{\gamma_0^2 - 4\delta_I^2}}{2} \tag{24}$$

One can check that the classical Hamiltonian, though complex, is symmetric under parity-time reversal. Furthermore, for $\gamma_0 \geq | 2\delta_I |$, $E$ is real, corresponding to exact or unbroken $\mathcal{PT}$ symmetry. However, for $\gamma_0 < | 2\delta_I |$, energies turn complex. Thus the classical system undergoes a phase transition at $\gamma_0 = | 2\delta_I |$. The corresponding quantum version, too, undergoes an abrupt *phase transition* from exact $\mathcal{PT}$ to spontaneously broken $\mathcal{PT}$ *phase* at some critical value of the coupling parameter $\delta_I$. Our aim in this work is to see whether this *phase transition* gets manifested in the trajectories traced out by the classical particle.

**Case 3 :** $\delta$ is complex; $\delta = \delta_R + i\delta_I$ (say)
It is easy to observe that the potential (17) is no longer $\mathcal{PT}$ symmetric; nor is it $\eta$-pseudo Hermitian, and the energy spectrum is, in general, complex. Naturally, it does not arouse our interest; so we shall not pursue this case any further.

### 3.1 Classical Trajectories

To plot the classical orbits, we need to derive the expressions for the trajectories of the classical particle in the complex plane, under the influence of the potential (17). For this purpose we assume the most general form of $c(E)$, viz., $c(E) = c_R(E) + ic_I(E)$
Putting $\delta = \delta_R + i\delta_I$ in the expression for $A^{\pm}$, i.e., $A^{\pm} = c(E)e^{\pm i\left(\theta_0+\alpha_0\sqrt{-E}t\right)}$, we obtain

$$
\begin{aligned}
&\mp ip\cosh\frac{\alpha_0 x}{2} + \sqrt{-E}\sinh\frac{\alpha_0 x}{2} + \frac{\delta_R + i\delta_I}{\sqrt{-E}} \\
&= c_R(E)\cos\left(\theta_0 + \alpha_0\sqrt{-E}t\right) \mp c_I(E)\sin\left(\theta_0 + \alpha_0\sqrt{-E}t\right) \\
&\quad + i\left\{c_I(E)\cos\left(\theta_0 + \alpha_0\sqrt{-E}t\right) \pm c_R(E)\sin\left(\theta_0 + \alpha_0\sqrt{-E}t\right)\right\}
\end{aligned}
\tag{25}
$$

and consequently, the final form of the trajectory $x(t)$ and momenta $p(t)$ as

$$
x(t) = \frac{2}{\alpha_0}\sinh^{-1}\left\{\frac{\delta_R - c_R(E)\sqrt{-E}\cos\left(\theta_0 + \alpha_0\sqrt{-E}t\right) \pm c_I(E)\sqrt{-E}\sin\left(\theta_0 + \alpha_0\sqrt{-E}t\right)}{E}\right\}
\tag{26}
$$

$$
p(t) = \frac{E\ \left\{-c_R(E)\sin\left(\theta_0 + \alpha_0\sqrt{-E}t\right) \pm c_I(E)\cos\left(\theta_0 + \alpha_0\sqrt{-E}t\right) \pm \frac{\delta_I}{\sqrt{-E}}\right\}}{\left[E^2 + \left\{\delta_R - c_R(E)\sqrt{-E}\cos\left(\theta_0 + \alpha_0\sqrt{-E}t\right) \pm c_I(E)\sqrt{-E}\sin\left(\theta_0 + \alpha_0\sqrt{-E}t\right)\right\}^2\right]^{1/2}}
\tag{27}
$$

Equation (26) describes the position and equation (27) the momentum of a classical particle in the complex plane, as it is under the influence of complex forces. $x(t)$ and $p(t)$ given above are complex, and therefore we shall plot the real and imaginary parts of the position and momentum separately, to gain an insight into the behaviour of the particle's motion. In this work we shall primarily concentrate on purely imaginary values of $\delta = i\delta_I$, to see if any connection exists between the reality of the spectrum and regularity of the classical orbits.

### 3.2 $\mathcal{PT}$ symmetric Scarf II potential : $\delta \; = \; i \; \delta_I$

Initially we start with bound states with real negative energies, in the unbroken $\mathcal{PT}$ *phase* in the quantum version. The corresponding classical condition for real energies is $2 \mid \delta_I \mid < \gamma_0$. To plot the exact trajectories it is essential to have a knowledge about the initial condition $x(0)$. The classical turning points are given by the roots of the equation $E - V(x) = 0$, for which we need the precise value of $E$. The turning points are obtained in the form $x_{\pm} = \pm a + ib$, and hence are symmetric with respect to the imaginary axis. This is expected as the classical turning points of a complex $\mathcal{PT}$-symmetric potential for real energies, $E$, essentially occur as $(z, -z^*) : (-a + ib, a + ib)$ [9]. We plot the orbits traced out by the classical particle in the complex plane, in Fig. 1, for parameter values $\alpha_0 = 2$ , $\delta = 2i$ , $\gamma_0 = 6$. Equation (24) suggests that classical energies should lie between $-0.763932$ and $-5.23607$. We plot the trajectories for $E = -3$. Any other value of energy within the specified range gives similar results. The classical turning points for this particular set of parameters are $\{\pm 0.781368 - 0.528945i\}$, $\{\pm 0.781368 - 2.61265i\}$, etc., i.e. of the type $(z_1, -z_1^*)$ , $(z_2, -z_2^*)$ , etc.

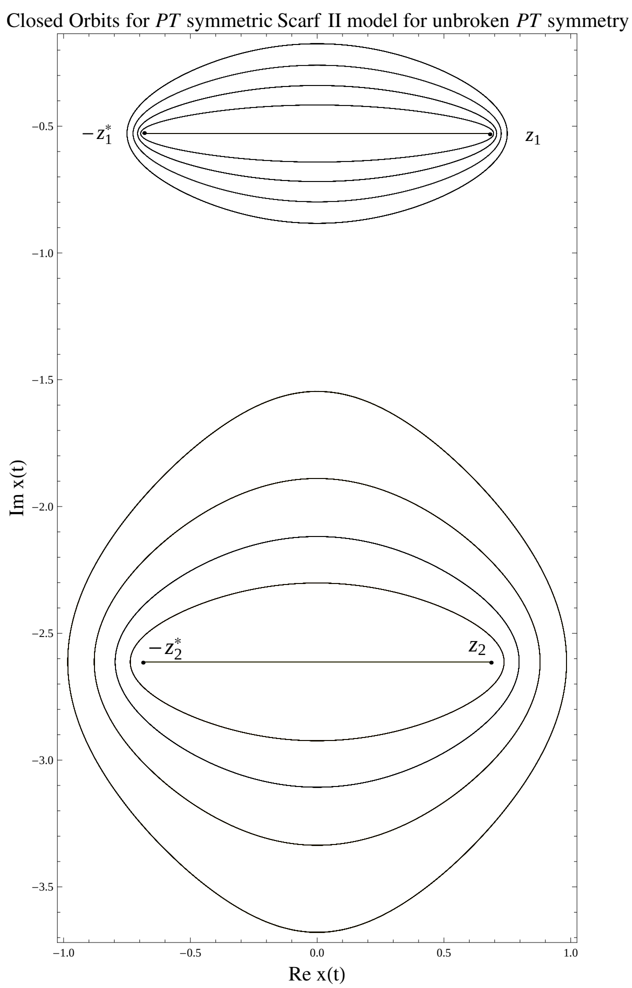


Figure 1: The classical trajectories for parameter values $E = -3$ , $\alpha_0 = 2$ , $\delta = 2i$ , $\gamma_0 = 6$

In case the classical particle begins its motion on the real line, within the turning points or from any one of the turning points itself, say $z_1$, the particle can only oscillate between the turning points $(z_1, -z_1^*)$. Similar is the picture for the particle starting from any position on the real line within the turning points $(z_2, -z_2^*)$, or from any one of them. These paths are shown by the horizontal lines in Fig. 1. A trajectory joining any other pair of turning points (e.g. $z_1, -z_2^*$ or $z_1, z_2$ etc.) is forbidden because the particle is under the influence of a $\mathcal{PT}$ symmetric potential. However, if the particle starts its motion from any other point in the complex plane, the oscillatory trajectories are surrounded by closed orbits of a definite periodicity. As seen in Fig. 1, the trajectories are closed curves, which do not cross. The time period for each of the orbits in Fig. 1 is calculated to be 3.6276. Thus our observations for this exactly solvable analytical model are similar to those obtained in ref. [**?**, 2] by numerical methods. In Fig. 2, we plot the momenta of the classical particle in the complex plane, and in Fig. 3 we plot the phase space curves, for the same set of parameter values as in Fig. 1. In these cases, depending on the initial starting point, the momenta plots as well as the phase space plots are closed curves without any crossings. However, while the trajectory plots and the phase space curves are symmetrical with respect to the imaginary axis, the momenta curves are symmetric with respect to the real axis.

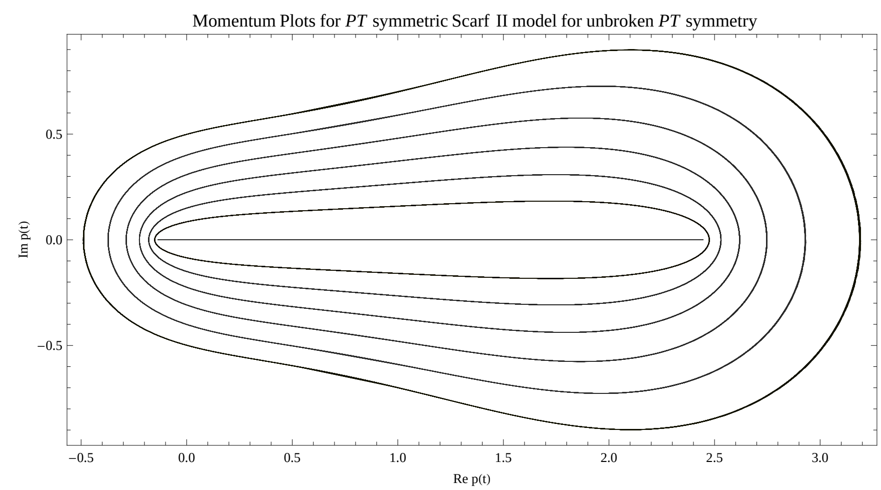


Figure 2: The classical momenta for parameter values $E = -3$ , $\alpha_0 = 2$ , $\delta = 2i$ , $\gamma_0 = 6$

Now if the parameter values are such that $2 \mid \delta_I \mid > \gamma_0$ , then equation (24) suggests that the energies turn out to be complex conjugate pairs, in spite of the Hamiltonian being $\mathcal{PT}$ symmetric. This is referred to as the spontaneously broken $\mathcal{PT}$ symmetric *phase* in the quantum version. The classical trajectories are plotted in Fig. 4, for parameter values $\alpha_0 = 2$ , $\delta = 2i$ , $\gamma_0 = 3$. From eq. (16), the energies for this set of parameter values should lie between $-1.5 + 1.32288i$ and $-1.5 - 1.32288i$. We have taken $E = -1.5 - 0.3i$ for the plots in Fig. 4. The abrupt *phase transition* in the quantum version gets manifested in the classical motion, too. Irrespective of the starting point of motion, all of a sudden the closed orbits become open and the trajectory loses its periodicity. The turning points are no longer of the type $(z, -z^*)$, and symmetry is lost. For this particular set of parameters the turning points occur at $(-0.102199 - 0.470998i$ , $0.102199 - 2.67059i$ , $-1.40526 - 1.3489i$ , $1.40526 - 1.79269i)$

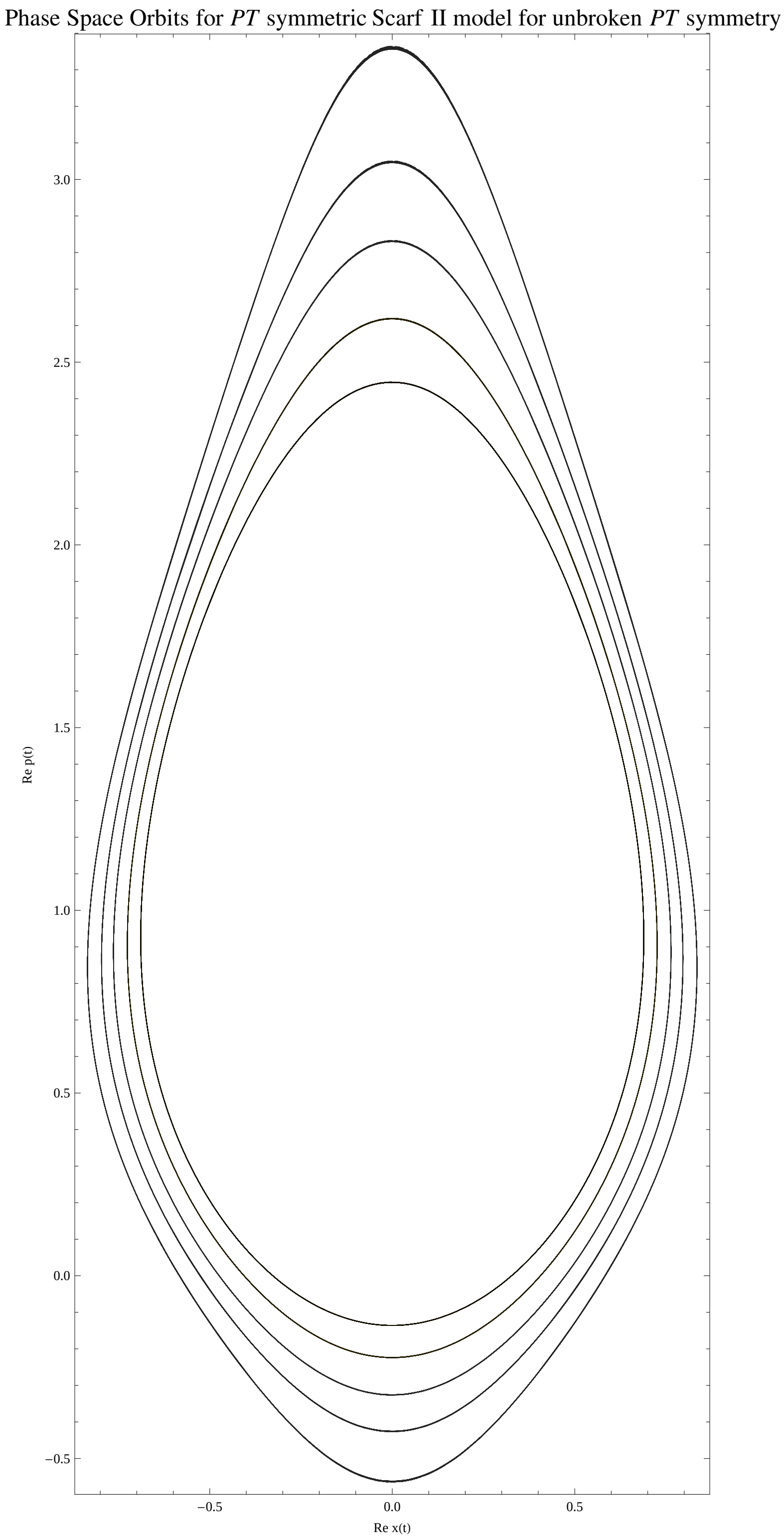


Figure 3: The real part of phase space trajectories for parameter values $E = -3$ , $\alpha_0 = 2$ , $\delta = 2i$ , $\gamma_0 = 6$

etc, i.e. are of the form $(a - ib_1\ ,\ -a - ib_2\ , c - id_1\ , -c - id_2)$. The momenta plots and phase space trajectories, too, lose their regular behaviour — the curves are no longer closed ones, nor do they show any definite periodicity.

## 4 Conclusions

To conclude, we have extended the factorization technique of ref. [1] to study complex classical systems. As an illustration we have applied the technique to study the exactly solvable classical analogue of the exactly solvable, non Hermitian yet $\mathcal{PT}$ symmetric Scarf II model

$$V(x) = -v_1 \ \mathrm{sech}^2 x - i\ v_2 \ \mathrm{sech}\ x\ \tanh x \qquad ; \qquad v_1 > 0,\ v_1 \ \text{real}$$

Since the particle is under the influence of complex forces, it is expected to move about in the complex $x$ plane. The quantum version of this particular Hamiltonian exhibits an abrupt *phase transition* at some critical value of $v_2$, switching energy values from real to complex conjugate pairs. On the other hand, for complex values of $v_2$, there is no space-time reflection symmetry, and energies are always complex. We have given special emphasis on real values of $v_2$ in this work, so as to study the effect of spontaneous $\mathcal{PT}$ symmetry breaking on the classical trajectories and momenta. Employing the factorization technique of ref. [1], we have found the equation of the path the classical particle traces out in the complex $x$ plane, and plotted the corresponding orbits in Fig. 1 and Fig. 4. It is observed that so long as $\mathcal{PT}$ symmetry is unbroken, the motion of the corresponding classical particle is bounded, with all trajectories having the same periodicity — Fig. 1. Furthermore, the orbits are symmetric with respect to the imaginary axis, and the classical turning points are of the form $(z, -z^*)$. The spontaneous breakdown of $\mathcal{PT}$ symmetry resulting in switching of energy values from real to complex conjugate pairs, has an interesting manifestation in the corresponding classical picture — the closed periodic orbits abruptly become irregular and open, as shown in Fig. 4. However, none of the trajectories cross each other, as is expected for systems symmetric under $\mathcal{PT}$.

We have also plotted the momenta of the classical particle in the complex plane in Fig. 2, and the phase space trajectories in Fig. 3, so long as $\mathcal{PT}$ symmetry is unbroken and energies are real. Here, too, the curves are closed, and of definite periodicity. While the trajectory plots and phase space plots are symmetric with respect to the imaginary axis, the momenta plots are symmetric with respect to the real axis. At the onset of spontaneous breakdown of $\mathcal{PT}$ symmetry at the exceptional point, each of these plots abruptly loses its regular pattern, and the closed curves become open.

To summarize, the observed deviation from regularity in the classical orbits at an exceptional point, for the analytical model considered in this work, is similar to that shown in earlier works [2, 6]. However, we follow the factorization approach given in [1] and thus our technique is different from the numerical / perturbative studies done in ref. [2, 6]. Additionally, the model considered here is exactly solvable, both in its quantum as well as classical versions.

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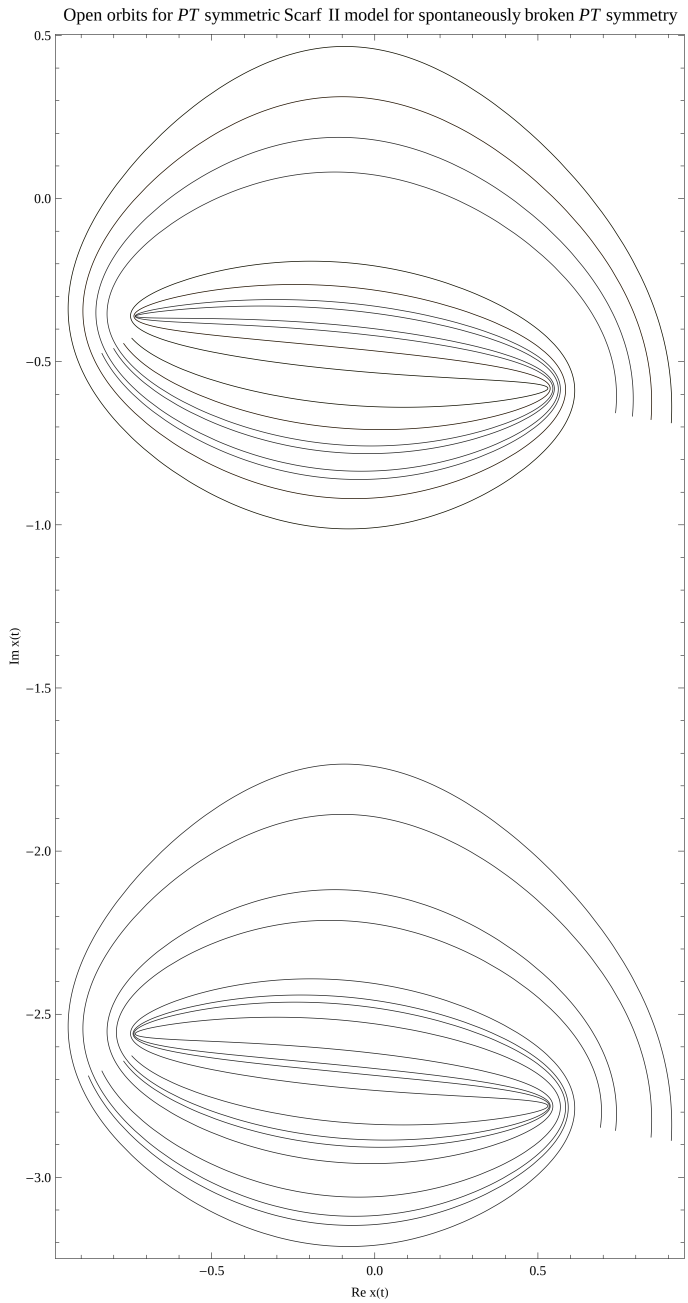


Figure 4: The classical trajectories for parameter values $E = -1.5 - .3i$ , $\alpha_0 = 2$ , $\delta = 2i$ , $\gamma_0 = 3$

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